MATH 1650: SECTION 3.2: GRAPHS OF RATIONAL FUNCTIONS SUMMARY

1. **DOMAIN:** Set the denominator equal to 0 to find the **excluded values**.
2. **REDUCE:** Reduce to lowest terms: **factor** numerator and denominator to cancel common **factors**.
3. **BEHAVIOR NEAR EXCLUDED VALUES:** If a factor corresponding to an excluded value **cancels** from the denominator, the graph has a **hole** there. Otherwise, the graph has a **vertical asymptote** there.
4. **INTERCEPTS:** Find the x- and y-intercepts, if any. Using the **reduced** form of the function:
   - find the x-intercept(s) by solving $y =$ the function $= 0$. Check that your answers are in the domain of the function. Remember: x-intercepts are **points**, they are of the form $(c, 0)$.
   - find the y-intercept by substituting $x = 0$ into the function (provided $x = 0$ is in the domain!)
5. **END BEHAVIOR:** Using the **reduced** form of the function:
   - **HORIZONTAL ASYMPTOTES** occur in one of two cases:
     - If the degree of the numerator is **less** than the degree of the denominator, the HA is $y = 0$.
     - If the degree of the numerator is the **same** as the degree of the denominator, the HA is:

$$y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$$

   - **SLANT (OBLIQUE) ASYMPTOTES** occur if the degree of the numerator is **exactly one more** than the degree of the denominator. To find the asymptote, you need to perform **long division**. The asymptote is the **quotient** of the division.
6. **SIGN DIAGRAMS** are useful as is plotting additional points to finish the graph.
   - To make a Sign Diagram for a Rational Function:
     - plot the **excluded** values on the number line with a '?' above them.
     - plot the **zeros** on the number line with a '0' above them.
   - Use the **reduced** form of the function to determine where the function is $(+)$ or $(-)$ using test values.

EXAMPLE: Let $f(x) = \dfrac{2x-3}{x+2}$.

- Find the values excluded from the domain of f.

  To find the excluded values, we set the denominator equal to 0. We solve $x+2=0$ and find $x=-2$.

- Write the domain of f using interval notation.

  The domain of f is $\{x \,|\, x \neq -2\}$ which in interval notation is $(-\infty,-2)\cup(-2,\infty)$.

- Reduce $f(x)$ to lowest terms.

  $f(x) = \dfrac{2x-3}{x+2}$ is already in lowest terms since neither the numerator nor denominator can be factored.

- Analyze the behavior of f near the excluded values.

  Since the factor $(x+2)$ didn't cancel from the denominator, $x=-2$ is a vertical asymptote.

- Find the axis intercepts, if any.

  To find the x-intercept, we set $y = f(x) = \dfrac{2x-3}{x+2} = 0$. We get $2x-3=0$ or $x=\frac{3}{2}$.

  The x-intercept is thus $\left(\frac{3}{2}, 0\right)$.

  To find the y-intercept, we set $x=0$ and get $y=f(0)=-\frac{3}{2}$. Hence the y-intercept is $\left(0,-\frac{3}{2}\right)$.

- Discuss the end behavior of the graph of $y=f(x)$.

  As $x\to\pm\infty$, $f(x) = \dfrac{2x-3}{x+2} \approx \dfrac{2x}{x} = 2$, so $y=2$ is a Horizontal Asymptote.

- Make a Sign Diagram for $f(x)$.

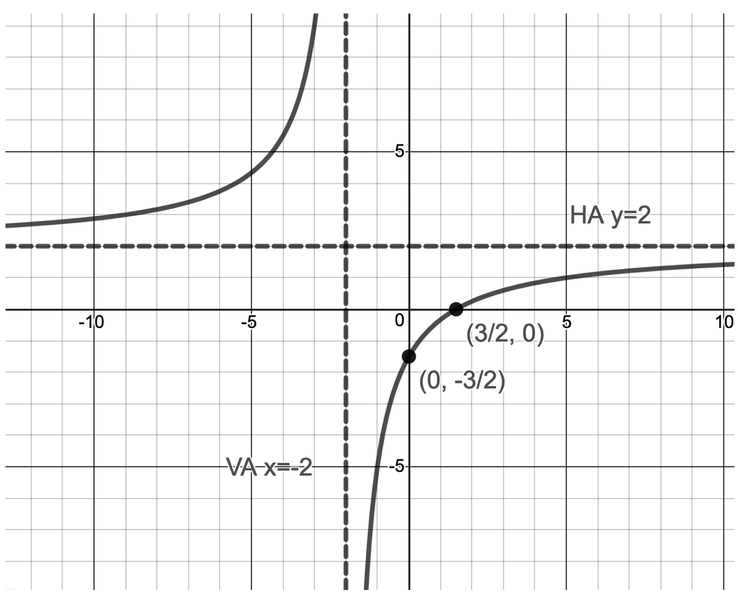
  We place the excluded value $x=-2$ with a '?' over it as well as $x=\frac{3}{2}$ with a '0' above it. Choosing test values $x=-3$, $x=0$, and $x=2$ gives us the Sign Diagram below.

$$\begin{array}{ccccccc} (+) & ? & (-) & 0 & (+) & & f(x) \\ \longleftarrow & + & \longrightarrow & + & \longrightarrow & & \\ & -2 & & \frac{3}{2} & & & x \end{array}$$

- Graph $y=f(x)$.

  We dash in the asymptotes, $x=-2$ and $y=2$, and plot the intercepts $\left(\frac{3}{2},0\right)$ and $\left(0,-\frac{3}{2}\right)$.

  We use the Sign Diagram as a guide as to how to fill in the rest.



EXAMPLE: Let $f(x)=\dfrac{x^2-2x-3}{1-x^2}$.

- Find the values excluded from the domain of f.

  We set the denominator equal to zero. Solving $1-x^2=0$, we get $x^2=1$ so $x=\pm1$.

- Write the domain of f using interval notation.

  The domain of f is $\{x\,|\,x\neq\pm1\}$ which in interval notation is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

- Reduce $f(x)$ to lowest terms.

$$f(x)=\frac{x^2-2x-3}{1-x^2}=\frac{(x-3)(x+1)}{-(x^2-1)}=\frac{(x-3)(x+1)}{-(x-1)(x+1)}=\frac{(x-3)\cancel{(x+1)}}{-(x-1)\cancel{(x+1)}}=\frac{x-3}{1-x}$$

  Hence, $f(x)=\dfrac{x-3}{1-x}$ provided $x\neq-1$.

- Analyze the behavior of f near the excluded values.

  Since the factor $(x+1)$ canceled from the denominator, we have a hole when $x=-1$.

  When $x\approx-1$, $f(x)\approx\dfrac{(-1)-3}{1-(-1)}=\dfrac{-4}{2}=-2$. Hence, there is a hole in the graph at $(-1,-2)$.

  Since the factor $(1-x)$ remains in the denominator, $x=1$ is a vertical asymptote.

- Find the axis intercepts, if any.

  To find the x-intercept, we once again use the reduced formula for $f(x)$ set $y=f(x)=\dfrac{x-3}{1-x}=0$.

  We get $x-3=0$ so $x=3$. The x-intercept is $(3,0)$.

  To find the y-intercept, we set $x=0$ in the reduced formula $f(x)=\dfrac{x-3}{1-x}$ and get $y=f(0)=\frac{-3}{1}=-3$. Hence the y-intercept is $(0,-3)$.

- Discuss the end behavior of the graph of $y=f(x)$.

  Once again, using the reduced formula for $f(x)$, we find as $x\to\pm\infty$, $f(x)=\dfrac{x-3}{1-x}\approx\dfrac{x}{-x}=-1$.

  Hence, $y=-1$ is a Horizontal Asymptote.

- Make a Sign Diagram for $f(x)$.

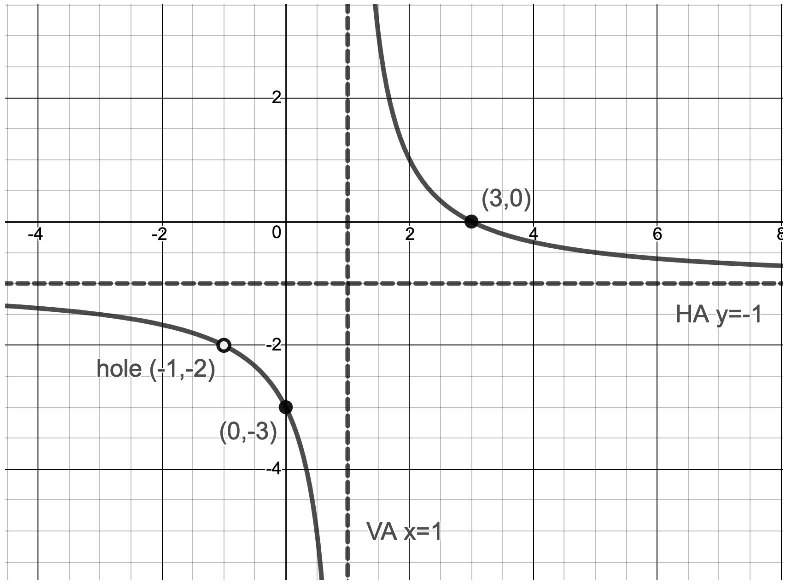
  We place the excluded values $x=\pm1$ with a '?' over it as well as $x=3$ with a '0' above it.

  Testing the sign of $f(x)$ around these values, we get:

$$\begin{array}{ccccccccc} (-) & ? & (-) & ? & (+) & 0 & (-) & & f(x) \\ \hline & -1 & & 1 & & 3 & & & x \end{array}$$

- Graph $y = f(x)$.

  We dash in the asymptotes, $x = 1$ and $y = -1$, and plot the intercepts $(3, 0)$ and $(0, -3)$ and the hole at $(-1, 2)$. We use the Sign Diagram as a guide as to how to fill in the rest.



EXAMPLE: Let $f(x) = \dfrac{4-x^2}{x+3}$.

- Find the values excluded from the domain of f.

  To find the excluded values, we set the denominator equal to 0. We solve $x+3=0$ and find $x=-3$.

- Write the domain of f using interval notation.

  The domain of f is $\{x \mid x \neq -3\}$ which in interval notation is $(-\infty,-3)\cup(-3,\infty)$.

- Reduce $f(x)$ to lowest terms.

  $f(x) = \dfrac{4-x^2}{x+3} = \dfrac{-(x-2)(x+2)}{x+3}$ is already in lowest terms since nothing cancels.

- Analyze the behavior of f near the excluded values.

  Since the factor $(x+3)$ didn't cancel from the denominator, $x=-3$ is a vertical asymptote.

- Find the axis intercepts, if any.

  To find the y-intercept, we set $x=0$ and get $y=f(0)=\frac{4}{3}$. Hence the y-intercept is $\left(0,\frac{4}{3}\right)$.

  To find the x-intercept, we set $y=f(x)=\dfrac{4-x^2}{x+3}=0$. We get $4-x^2=0$ so $x^2=4$ so $x=\pm 2$.

  The x-intercepts are $(-2,0)$ and $(2,0)$.

- Discuss the end behavior of the graph of $y=f(x)$.

  As $x\to\pm\infty$, $f(x)=\dfrac{4-x^2}{x+3}\approx\dfrac{-x^2}{x}=-x$, so we have a slant asymptote.

  In this case we can use long or synthetic division to get: $f(x)=\dfrac{4-x^2}{x+3}=-x+3-\dfrac{5}{x+3}$.

  The slant asymptote is $y=-x+3$.

- Make a Sign Diagram for $f(x)$.

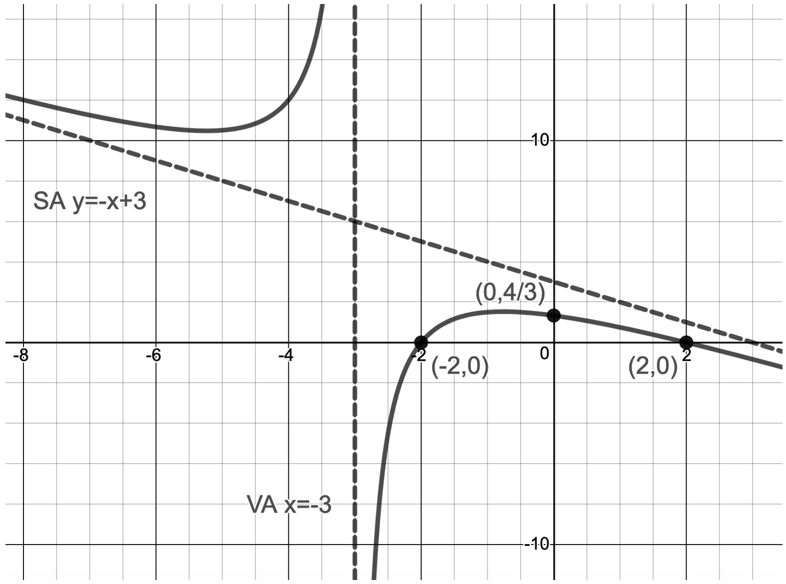
  We place the excluded value $x=-3$ with a '?' over it as well as $x=\pm 2$ with a '0' above them. We get:

$$\begin{array}{ccccccccc} (+) & ? & (-) & 0 & (+) & 0 & (-) & & f(x) \\ \longleftarrow & -3 & \text{---} & -2 & \text{---} & 2 & \longrightarrow & & x \end{array}$$

- Graph $y = f(x)$.

  We dash in the asymptotes, $x = -3$ and $y = -x + 3$, and plot the intercepts $(\pm 2, 0)$ and $\left(0, \frac{4}{3}\right)$.

  We use the Sign Diagram as a guide as to how to fill in the rest.



EXAMPLE: Let $f(x) = \dfrac{x^2 + 6x + 9}{x^2 - 9}$.

- Find the values excluded from the domain:

- State the domain using interval notation:

- Reduce $f(x)$ to lowest terms.

- List the vertical asymptotes, if any:

- Find the holes in the graph, if any:

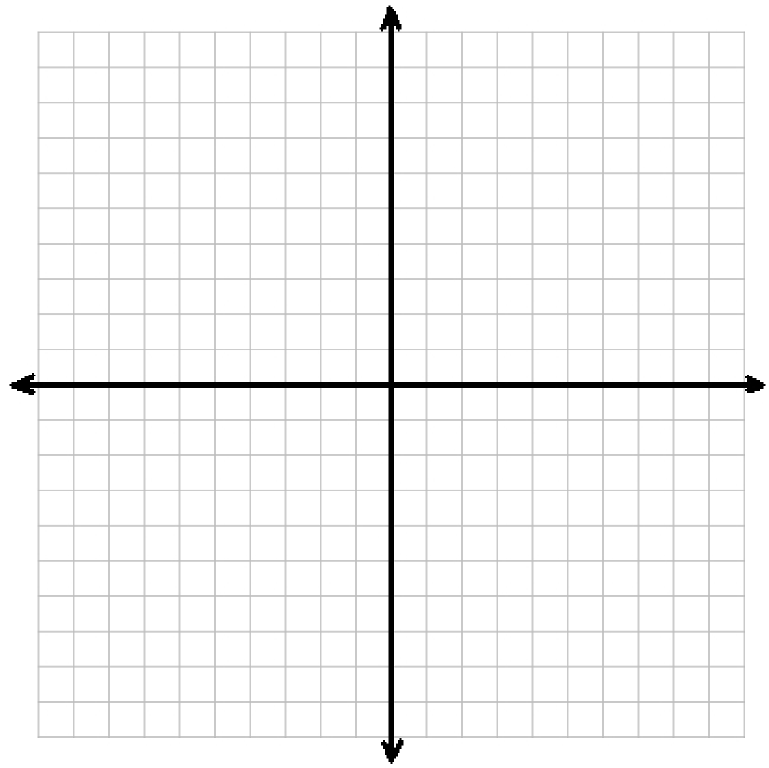
- Find the y-intercept, if any.

- Find the x-intercepts, if any.

- Find the horizontal or slant asymptote, if any.

- Make a Sign Diagram for $f(x)$.

- Graph $y = f(x)$.



HOMEWORK: Section 3.2: 1 - 19 odd.